(Anti)symmetric multivariate trigonometric functions and corresponding Fourier transforms

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Abstract. Four families of special functions, depending on n variables, are studied. We call them symmetric and antisymmetric multivariate sine and cosine functions. They are given as determinants or antideterminants of matrices, whose matrix elements are sine or cosine functions of one variable each. These functions are eigenfunctions of the Laplace operator, satisfying specific conditions at the boundary of a certain domain F of the n-dimensional Euclidean space. Discrete and continuous orthogonality on F of the functions within each family, allows one to introduce symmetrized and antisymmetrized multivariate Fourier-like transforms, involving the symmetric and antisymmetric multivariate sine and cosine functions.

1. Introduction

In mathematical and theoretical physics, very often we deal with functions on the Euclidean space E_n which are symmetric or antisymmetric with respect to the permutation (symmetric) group S_n. For example, such functions describe collections of identical particles. Symmetric or antisymmetric solutions appear in the theory of integrable systems. Characters of finite dimensional representations of semisimple Lie algebras are symmetric functions. Moreover, according to the Weyl formula for these characters, each such character is ratio of antisymmetric functions.

The aim of this paper is to introduce, to describe and to study symmetrized and antisymmetrized multivariate sine and cosine functions and the corresponding Fourier transforms. Antisymmetric multivariate sine and cosine functions (we denote them by $\sin^-_\lambda(x)$, $\cos^-_\lambda(x)$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $x = (x_1, x_2, \dots, x_n)$) are determinants of $n \times n$ matrices, whose entries are sine or cosine functions of one variable,

$$\sin^-_\lambda(x) = \det(\sin 2\pi\lambda_i x_j)^n_{i,j=1}\,, \qquad \cos^-_\lambda(x) = \det(\cos 2\pi\lambda_i x_j)^n_{i,j=1}\,.$$

These functions are antisymmetric in variables $x_1, x_2, \dots, x_n$ and in parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. These antisymmetricities follow from antisymmetricity of a determinant of a matrix under permutation of rows or of columns.

Symmetric multivariate sine and cosine functions $\sin^+_\lambda(x)$, $\cos^+_\lambda(x)$ are antideterminants of the same $n \times n$ matrices (a definition of antideterminants see below). These functions are symmetric in variables and in parameters. This symmetricity follows from symmetricity of an antideterminant of a matrix under permutation of rows or of columns.

As in the case of sine and cosine functions of one variable, we may consider three types of symmetric and antisymmetric multivariate trigonometric functions:

(a) functions $\sin^\pm_m(x)$ and $\cos^\pm_m(x)$ such that $m=(m_1,m_2,\dots,m_n)$, $m_i\in\mathbb{Z}$; these functions determine series expansions in multivariate sine and cosine functions;

(b) functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$ such that $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$, $\lambda_i\in\mathbb{R}$; these functions determine sine and cosine integral Fourier transforms;

(c) functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$, where $x=(x_1,x_2,\dots,x_n)$ takes a finite set of values; these functions determine multivariate finite sine and cosine Fourier transforms.

Functions (b) are antisymmetric (symmetric) with respect to elements of the permutation group S_n. (Anti)symmetries of functions (a) are described by a wider group, since sine and cosine functions $\sin 2\pi mx$, $\cos 2\pi mx$, $m\in\mathbb{Z}$, are invariant with respect to shifts $x\to x+k$, $k\in\mathbb{Z}$. (Anti)symmetries of functions (a) are described by elements of the extended affine symmetric group $\tilde S^{\rm aff}_n$ which is a product of the groups S_n, T_n and Z_2^n, where T_n consists of shifts of E_n by vectors $r=(r_1,r_2,\dots,r_n)$, $r_j\in\mathbb{Z}$, and Z_2^n is a product of n copies of the group Z_2 of order 2 generated by the reflection with respect to the zero point. A fundamental domain $F(\tilde S^{\rm aff}_n)$ of the group $\tilde S^{\rm aff}_n$ is a certain bounded set in $\mathbb{R}^n$.

The functions $\cos^+_m(x)$ with $m=(m_1,m_2,\dots,m_n)$, $m_i\in\mathbb{Z}$, give solutions of the Neumann boundary value problem on a closure of the fundamental domain $F(\tilde S^{\rm aff}_n)$. The functions $\sin^-_m(x)$ with $m=(m_1,m_2,\dots,m_n)$, $m_i\in\mathbb{Z}$, are solutions of the Laplace equation $\Delta f=\mu f$ on the domain $F(\tilde S^{\rm aff}_n)$ vanishing on the boundary $\partial F(\tilde S^{\rm aff}_n)$ of $F(\tilde S^{\rm aff}_n)$ (Dirichlet boundary problem). The functions (b) are also solutions of the Laplace equation.

Functions on the fundamental domain $F(\tilde S^{\rm aff}_n)$ can be expanded into series in the functions (a). These expansions are an analogue of the usual sine and cosine Fourier series for functions of one variable. Functions (b) determine antisymmetrized and symmetrized sine and cosine Fourier integral transforms on the fundamental domain $F(\tilde S_n)$ of the extended symmetric group $\tilde S_n=S_n\times Z_2^n$. This domain consists of points $x\in E_n$ such that $x_1>x_2>\dots>x_n>0$.

Functions (c) are used to determine antisymmetric or symmetric finite (that is, on a finite set) trigonometric multivariate Fourier transforms. These finite trigonometric transforms are given on grids consisting of points of the fundamental domain $F(\tilde S^{\rm aff}_n)$.

Symmetric and antisymmetric sine and cosine functions, which are studied in this paper, are closely related to symmetric and antisymmetric orbit functions defined in Refs. 1 and 2 and studied in detail in Refs. 3 and 4. They are connected with orbit functions corresponding to the Dynkin–Coxeter diagrams of semisimple Lie algebras of rank n. Discrete orbit function transforms, corresponding to Dynkin–Coxeter diagrams of low rank, were studied and exploited in rather useful applications[5−13]. Clearly, our multivariate sine and cosine transforms can be applied under solution of the same problems, that is, of the problems formulated on grids or lattices.

The exposition of the theory of orbit functions in Refs. 3 and 4 strongly depends on the theory of Weyl groups, properties of root systems, etc. In this paper we avoid this dependence (moreover, some of our functions cannot be treated by using Weyl gorups and root systems). We use only the permutation (symmetric) group, its extension, and properties of determinants and antideterminants.

It is well-known that the determinant $\det(a_{ij})_{i,j=1}^n$ of an $n\times n$ matrix $(a_{ij})_{i,j=1}^n$ is defined as

$$\det(a_{ij})_{i,j=1}^n = \sum_{w\in S_n} (\det w) a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} \\ = \sum_{w\in S_n} (\det w) a_{w(1),1} a_{w(2),2} \cdots a_{w(n),n}, \qquad (1)$$

where S_n is the permutation (symmetric) group of n symbols $1, 2, \dots, n$, the set $(w(1), w(2), \dots,\ w(n))$ means the set $w(1, 2, \dots, n)$, and $\det w$ denotes a determinant of the transform w, that is, $\det w = 1$ if w is an even permutation and $\det w = -1$ otherwise. Along with a determinant, we use an antideterminant $\det^+$ of the matrix $(a_{ij})_{i,j=1}^n$ which is defined as a sum of all summands entering into the expression for a determinant, taken with the sign $+$,

$$\det{}^+(a_{ij})_{i,j=1}^n = \sum_{w\in S_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} \\ = \sum_{w\in S_n} a_{w(1),1} a_{w(2),2} \cdots a_{w(n),2}. \qquad (2)$$

Symmetrized or antisymmetrized multivariate sine and cosine functions were mentioned in Refs. 14 and 15. In this paper we investigate in detail these multivariate functions and derive the corresponding continuous and finite Fourier transforms. Note that in Ref. 16 we have studied symmetric and antisymmetric exponential functions.

2. Symmetric and antisymmetric multivariate sine and cosine functions

Antisymmetric multivariate sine functions $\sin_\lambda^-(x)$ on $\mathbb{R}^n$ are determined by n real numbers $\lambda \equiv (\lambda_1, \lambda_2, \dots, \lambda_n)$ and are given by the formula

$$\begin{aligned} \sin_\lambda^-(x) \equiv \sin^-_{(\lambda_1,\lambda_2,\dots,\lambda_n)}(x_1, x_2, \dots, x_n) &:= \det\left(\sin 2\pi\lambda_i x_j\right)_{i,j=1}^n \\ &= \det \begin{pmatrix} \sin 2\pi\lambda_1 x_1 & \sin 2\pi\lambda_1 x_2 & \cdots & \sin 2\pi\lambda_1 x_n \\ \sin 2\pi\lambda_2 x_1 & \sin 2\pi\lambda_2 x_2 & \cdots & \sin 2\pi\lambda_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ \sin 2\pi\lambda_n x_1 & \sin 2\pi\lambda_n x_2 & \cdots & \sin 2\pi\lambda_n x_n \end{pmatrix} \\ &\equiv \sum_{w\in S_n} (\det w) \sin 2\pi\lambda_1 x_{w(1)} \sin 2\pi\lambda_2 x_{w(2)} \cdots \sin 2\pi\lambda_n x_{w(n)} \\ &= \sum_{w\in S_n} (\det w) \sin 2\pi\lambda_{w(1)} x_1 \sin 2\pi\lambda_{w(2)} x_2 \cdots \sin 2\pi\lambda_{w(n)} x_n, \end{aligned} \qquad (3)$$

where $(w(1), w(2), \dots,\ w(n))$ means the set $w(1, 2, \dots, n)$ and $\det w$ denotes a determinant of the transform w, $\det w = \pm 1$.

A special case of the antisymmetric multivariate sine functions is when λ_i are integers; in this case we write $(m_1, m_2, \dots, m_n)$ instead of $(\lambda_1, \lambda_2, \dots, m_n)$,

$$\sin^-_{(m_1,m_2,\dots,m_n)}(x) = \det\left(\sin 2\pi m_i x_j\right)_{i,j=1}^n, \qquad m\in\mathbb{Z}.$$

If $n = 2$ we have

$$\begin{aligned}\sin^-_{(\lambda_1,\lambda_2)}(x_1,x_2) =& \sin\pi\lambda_1 x_1\ \sin\pi\lambda_2 x_2 - \sin\pi\lambda_1 x_2\ \sin\pi\lambda_2 x_1\\ =& \frac{1}{2}\cos 2\pi(\lambda_1 x_1 - \lambda_2 x_2) - \frac{1}{2}\cos 2\pi(\lambda_1 x_1 + \lambda_2 x_2)\\ &- \frac{1}{2}\cos 2\pi(\lambda_1 x_2 - \lambda_2 x_1) + \frac{1}{2}\cos 2\pi(\lambda_1 x_2 + \lambda_2 x_1).\end{aligned}$$

Antisymmetric multivariate cosine functions on $\mathbb{R}^n$ are determined by n real numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and are given by the formula

$$\begin{aligned}\cos^-_\lambda(x) \equiv& \cos^-_{(\lambda_1,\lambda_2,\dots,\lambda_n)}(x_1,x_2,\dots,x_n) := \det\left(\cos 2\pi\lambda_i x_j\right)_{i,j=1}^n\\ \equiv& \sum_{w\in S_n}(\det w)\cos 2\pi\lambda_1 x_{w(1)}\cos 2\pi\lambda_1 x_{w(2)}\cdots\cos 2\pi\lambda_1 x_{w(n)}\\ =& \sum_{w\in S_n}(\det w)\cos 2\pi\lambda_{w(1)}x_1\cos 2\pi\lambda_{w(2)}x_2\cdots\cos 2\pi\lambda_{w(n)}x_n.\end{aligned}\tag{4}$$

As in the case of sine functions, we separate the special case of the antisymmetric multivariate cosine functions when λ_i are integers and write $(m_1, m_2, \dots, m_n)$ instead of $(\lambda_1, \lambda_2, \dots, \lambda_n)$.

If $n = 2$, we get

$$\begin{aligned}\cos^-_{(\lambda_1,\lambda_2)}(x_1,x_2) =& \frac{1}{2}\cos 2\pi(\lambda_1 x_1 - \lambda_2 x_2) + \frac{1}{2}\cos 2\pi(\lambda_1 x_1 + \lambda_2 x_2)\\ &- \frac{1}{2}\cos 2\pi(\lambda_1 x_2 - \lambda_2 x_1) - \frac{1}{2}\cos 2\pi(\lambda_1 x_2 + \lambda_2 x_1).\end{aligned}$$

Symmetric multivariate sine functions on $\mathbb{R}^n$ are determined by n real numbers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and are given by

$$\begin{aligned}\sin^+_\lambda(x) \equiv& \sin^+_{(\lambda_1,\lambda_2,\dots,\lambda_n)}(x_1,x_2,\dots,x_n) := \det{}^+\left(\sin 2\pi\lambda_i x_j\right)_{i,j=1}^n\\ =& \sum_{w\in S_n}\sin 2\pi m_1 x_{w(1)}\cdot\ldots\cdot\sin 2\pi m_n x_{w(n)}\\ =& \sum_{w\in S_n}\sin 2\pi m_{w(1)}x_1\cdot\ldots\cdot\sin 2\pi m_{w(n)}x_n,\end{aligned}\tag{5}$$

If $n = 2$ we have

$$\begin{aligned}\sin^+_{(\lambda_1,\lambda_2)}(x_1,x_2) =& \frac{1}{2}\cos 2\pi(\lambda_1 x_1 - \lambda_2 x_2) - \frac{1}{2}\cos 2\pi(\lambda_1 x_1 + \lambda_2 x_2)\\ &+ \frac{1}{2}\cos 2\pi(\lambda_1 x_2 - \lambda_2 x_1) - \frac{1}{2}\cos 2\pi(\lambda_1 x_2 + \lambda_2 x_1).\end{aligned}$$

Symmetric multivariate cosine functions on $\mathbb{R}^n$ are determined by n real numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and are given by the formula

$$\begin{aligned}\cos^+_\lambda(x) =& \det{}^+\left(\cos 2\pi\lambda_i x_j\right)_{i,j=1}^n\\ =& \sum_{w\in S_n}\cos 2\pi m_1 x_{w(1)}\cdot\ldots\cdot\cos 2\pi m_n x_{w(n)}\\ =& \sum_{w\in S_n}\cos 2\pi m_{w(1)}x_1\cdot\ldots\cdot\cos 2\pi m_{w(n)}x_n.\end{aligned}\tag{6}$$

If $n=2$, we get

$$\begin{aligned}\cos^+_{(\lambda_1,\lambda_2)}(x_1,x_2) =& \frac{1}{2}\cos 2\pi(\lambda_1x_1-\lambda_2x_2)+\frac{1}{2}\cos 2\pi(\lambda_1x_1+\lambda_2x_2)\\ &+\frac{1}{2}\cos 2\pi(\lambda_1x_2-\lambda_2x_1)+\frac{1}{2}\cos 2\pi(\lambda_1x_2+\lambda_2x_1).\end{aligned}$$

A special case of the symmetric multivariate sine and cosine functions is when λ_i are integers; in this case we write $(m_1, m_2, \dots, m_n)$ instead of $(\lambda_1, \lambda_2, \dots, m_n)$.

3. Extended affine symmetric group and fundamental domains

It follows from the definitions of the functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$ that a permutation of variables $x_1, x_2, \dots, x_n$ is equivalent to the same permutation of the corresponding rows in the matrices from the definition. Then due to properties of determinants and antideterminants of matrices under permutations of rows, the functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$ are symmetric or antisymmetric with respect to the permutation group S_n, that is,

$$\sin^+_\lambda(wx)=\sin^+_\lambda(x),\qquad \cos^+_\lambda(wx)=\cos^+_\lambda(x),\qquad w\in S_n,$$

$$\sin^-_\lambda(wx)=(\det w)\sin^-_\lambda(x),\quad \cos^-_\lambda(wx)=(\det w)\cos^-_\lambda(x),\quad w\in S_n.$$

These functions admit additional (anti)invariances with respect to changing signs of coordinates $x_1, x_2, \dots, x_n$. Let ε_i denote the operation of a change of a sign of the coordinate x_i. Since $\sin 2\pi\varepsilon_i x_i\lambda_j = -\sin 2\pi x_i\lambda_j$ and $\cos 2\pi\varepsilon_i x_i\lambda_j = \cos 2\pi x_i\lambda_j$, we have

$$\sin^+_\lambda(\varepsilon_i x)=-\sin^+_\lambda(x),\quad \cos^+_\lambda(\varepsilon_i x)=\cos^+_\lambda(x), \tag{7}$$

$$\sin^-_\lambda(\varepsilon_i x)=-\sin^-_\lambda(x),\quad \cos^-_\lambda(\varepsilon_i x)=\cos^-_\lambda(x). \tag{8}$$

We denote the group generated by changes of coordinate signs by Z_2^n, where Z_2 is the group of changes of sign of one coordinate.

The group $\tilde{S}_n = S_n \times Z_2^n$ is called the *extended symmetric group*. It is a group of (anti)symmetries for the functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$.

We have the same (anti)symmetries under changes of signs in numbers $\lambda_1, \lambda_2, \dots, \lambda_n$. In order to avoid these (anti)symmetries, we may assume that all coordinates $x_1, x_2, \dots, x_n$ and all numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are non-negative.

The trigonometric functions $\sin^+_m$ and $\cos^+_m$, determined by integral $m=(m_1, m_2, \dots, m_n)$, admit additional symmetries related to periodicity of the sine and cosine functions $\sin 2\pi ry$, $\cos 2\pi ry$ of one variable for integral values of r. These symmetries are described by the discrete group of shifts by vectors

$$r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n,\quad r_i\in\mathbb{Z},$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the unit vectors in directions of the corresponding axes. We denote this group by T_n. Permutations of S_n, the operations ε_i of changes of coordinate signs, and shifts of T_n generate a group which is denoted as $\tilde{S}_n^{\rm aff}$ and is called the *extended affine symmetric group*. (The group generated by permutations of S_n and by shifts of T_n generate a group which is denoted as $S_n^{\rm aff}$ and is called the *affine symmetric group*). Thus, the group $\tilde{S}_n^{\rm aff}$ is a product of its subgroups,

$$\tilde{S}_n^{\rm aff} = S_n \times Z_2^n \times T_n,$$

where T_n is an invariant subgroup, that is, $wtw^{-1}\in T_n$ and $\varepsilon_i t\varepsilon_i^{-1}\in T_n$ for $w\in S_n$, $\varepsilon_i\in Z_2$, $t\in T_n$.

An open connected simply connected set $F \subset \mathbb{R}^n$ is called a *fundamental domain* for the group $\tilde{S}_n^{\rm aff}$ (for the group $\tilde{S}_n$) if it does not contain equivalent points (that is, points x and x' such that $x' = gx$, where g is an element of $\tilde{S}_n^{\rm aff}$ or $\tilde{S}_n$) and if its closure contains at least one point from each $\tilde{S}_n^{\rm aff}$-orbit (from each $\tilde{S}_n$-orbit). Recall that an $\tilde{S}_n^{\rm aff}$-orbit of a point $x \in \mathbb{R}^n$ is the set of points wx, $w \in \tilde{S}_n^{\rm aff}$.

It is evident that the set D_+^+ of all points $x = (x_1, x_2, \dots, x_n)$ such that

$$x_1 > x_2 > \cdots > x_n > 0 \tag{9}$$

is a fundamental domain for the group $\tilde{S}_n$ (we denote it as $F(\tilde{S}_n)$). The set of points $x = (x_1, x_2, \dots, x_n) \in D_+^+$ such that

$$\tfrac12 > x_1 > x_2 > \cdots > x_n > 0 \tag{10}$$

is a fundamental domain for the extended affine symmetric group $\tilde{S}_n^{\rm aff}$ (we denote it as $F(\tilde{S}_n^{\rm aff})$).

Remark. It may be seemed that instead of $\frac12$ in (10) there must be 1. However, in the group $\tilde{S}_n^{\rm aff}$ there exists the reflection of $\mathbb{R}^n$ with respect to the hyperplane $x_1 = \frac12$. This reflection coincides with the transform

$$x \to \varepsilon_1 x + \mathbf{e}_1,$$

where ε_1 is a change of a sign of the coordinate x_1 (note that ε_1 is the reflection of $\mathbb{R}^n$ with respect to the hyperplane $x_1 = 0$). The transform $x \to \varepsilon_1 x + \mathbf{e}_1$ leaves coordinates $x_2, x_3, \dots, x_n$ invariant and does not move the hyperplane $x_1 = \frac12$, that is, it is a reflection. Therefore, the domain $1 > x_1 > x_2 > \cdots > x_n > 0$ consists of two copies of the fundamental domain $F(\tilde{S}_n^{\rm aff})$.

In the group $\tilde{S}_n^{\rm aff}$ there also exist the reflections of $\mathbb{R}^n$ with respect to the hyperplanes $x_i = \frac12$, $i = 2, 3, \dots, n$.

As we have seen, the symmetric multivariate trigonometric functions $\sin_\lambda^+(x)$ and $\cos_\lambda^+(x)$ are symmetric with respect to the symmetric group S_n and behave according to formula (7) under a change of coordinate signs. This means that it is enough to consider these functions only on the closure of the fundamental domain $F(\tilde{S}_n)$, that is, on the set D_+ of points x such that

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge 0.$$

Values of these functions on other points are received by using symmetricity.

Symmetricity of $\sin_m^+$ and $\cos_m^+$ with integral $m = (m_1, m_2, \dots, m_n)$ with respect to the extended affine symmetric group $\tilde{S}_n^{\rm aff}$,

$$\sin_m^+(wx + r) = \sin_m^+(x), \quad \cos_m^+(wx + r) = \cos_m^+(x) \qquad w \in S_n, \quad r \in T_n, \tag{11}$$

$$\sin_m^+(\varepsilon_i x) = -\sin_m^+(x), \quad \cos_m^+(\varepsilon_i x) = \cos_m^+(x), \qquad \varepsilon_i \in Z_2, \tag{12}$$

means that we may consider the functions $\sin_m^+$ and $\cos_m^+$ on the closure of the fundamental domain $F(\tilde{S}_n^{\rm aff})$, that is, on the set of points

$$\tfrac12 \ge x_1 \ge x_2 \ge \cdots \ge x_n \ge 0.$$

Values of these functions on other points are obtained by using relations (11) and (12).

The functions $\sin_\lambda^-(x)$ and $\cos_\lambda^-(x)$ are antisymmetric with respect to the extended symmetric group $\tilde{S}_n$,

$$\sin_\lambda^-(wx) = (\det w)\sin_\lambda^-(x), \quad \cos_\lambda^-(wx) = (\det w)\cos_\lambda^-(x), \qquad w \in S_n,$$

$$\sin^-_\lambda(\varepsilon_i x) = -\sin^-_\lambda(x), \quad \cos^-_\lambda(\varepsilon_i x) = \cos^-_\lambda(x), \qquad \varepsilon_i \in Z_2.$$

For this reason, we may consider these functions only on the fundamental domain $F(\tilde{S}_n)$.

The antisymmetric sine and cosine functions $\sin^-_m(x)$ and $\cos^-_m(x)$ with integral $m = (m_1, m_2, \dots, m_n)$ also admit additional symmetries related to the periodicity of the usual sine and cosine functions. These symmetries are described by elements of the extended affine symmetric group $\tilde{S}_n^{\rm aff}$. For $w \in S_n$, $r \in T_n$ and $\varepsilon_i \in Z_2$ we have

$$\sin^-_m(wx + r) = (\det w)\sin^-_m(x), \quad \cos^-_m(wx + r) = (\det w)\cos^-_m(x), \tag{13}$$

$$\sin^-_m(\varepsilon_i x) = -\sin^-_m(x), \quad \cos^-_m(\varepsilon_i x) = \cos^-_m(x), \tag{14}$$

that is, it is enough to consider these functions only on the closure of the fundamental domain $F(\tilde{S}_n^{\rm aff})$. Values of these functions on other points are obtained by using the relations (13) and (14).

4. Properties

Symmetricity and antisymmetricity of symmetric and antisymmetric multivariate trigonometric functions is a main property of these functions. However, they possess many other interesting properties.

Behavior on boundary of fundamental domain. The symmetric and antisymmetric functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$ are finite sums of products of sine or cosine functions. Therefore, they are continuous functions of $x_1, x_2, \dots, x_n$ and have continuous derivatives of all orders on $\mathbb{R}^n$.

The boundary $\partial F(\tilde{S}_n)$ of the fundamental domain $F(\tilde{S}_n)$ consists of points of $F(\tilde{S}_n)$ which belong at least to one of the hyperplanes

$$x_1 = x_2, \quad x_2 = x_3, \quad \cdots, \quad x_{n-1} = x_n, \quad x_n = 0.$$

The set of points of the boundary, determined by the hyperplane $x_i = x_{i+1}$ or by the hyperplane $x_n = 0$, is called a wall of the fundamental domain.

Since for $x_i = x_{i+1}$, $i = 1, 2, \dots, n-1$, the matrix $(\sin 2\pi\lambda_i x_j)^n_{i,j=1}$ has two coinciding columns, then $\det(\sin 2\pi\lambda_i x_j)^n_{i,j=1} = 0$ in these cases. Clearly, we also have $\det(\sin 2\pi\lambda_i x_j)^n_{i,j=1} = 0$ for $x_n = 0$. Thus, *the function* $\sin^-_\lambda(x)$ *vanishes on the boundary* $\partial F(\tilde{S}_n)$.

It is shown similarly that the function $\cos^-_\lambda(x)$ vanishes on the walls $x_i = x_{i+1}$, $i = 1, 2, \dots, n-1$, of the boundary and $\partial\cos^-_\lambda(x)/\partial x_n$ vanishes on the wall $x_n = 0$.

The antideterminant $\det^+(\cos 2\pi\lambda_i x_j)^n_{i,j=1}$ does not change under permutation of two coinciding columns which appear on the walls $x_i = x_{i+1}$, $i = 1, 2, \dots, n-1$, of the boundary $\partial F(\tilde{S}_n)$. Moreover, the function $\cos^+_\lambda(x)$ is invariant under reflections r_i with respect to the hyperplanes $x_i = x_{i+1}$, $i = 1, 2, \dots, n-1$ (these reflections lead to permutations of the corresponding coordinates x_i and x_{i+1}). We also have $\partial \cos^+_\lambda(x)/\partial\, x_n = 0$ for $x_n = 0$. These assertions means that

$$\frac{\partial \cos^+_\lambda(x)}{\partial\, \mathbf{n}}\Big|_{\partial F(\tilde{S}_n)} = 0,$$

where $\mathbf{n}$ is the normal to the boundary $\partial F(\tilde{S}_n)$.

For $\sin^+_\lambda(x)$ we have that $\partial \sin^+_\lambda(x)/\partial\, \mathbf{n}$ vanishes on the walls $x_i = x_{i+1}$, $i = 1, 2, \dots, n-1$, and $\sin^+_\lambda(x) = 0$ on the wall $x_n = 0$.

Scaling symmetry. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. For $c \in \mathbb{R}$ we denote by $c\lambda$ the set $(c\lambda_1, c\lambda_2, \dots, c\lambda_n)$. We have

$$\sin^-_{c\lambda}(x) = \det(\sin 2\pi c\lambda_i x_j)^n_{i,j=1} = \sin^-_\lambda(cx).$$

This equality expresses the *scaling symmetry of antisymmetric sine functions.* It is shown similarly that

$$\cos^-_{c\lambda}(x) = \cos^-_\lambda(cx), \quad \sin^+_{c\lambda}(x) = \sin^+_\lambda(cx), \quad \cos^+_{c\lambda}(x) = \cos^+_\lambda(cx).$$

Duality. It follows from formulas for antisymmetric sine and cosine functions that they do not change under permutation $(\lambda_1, \lambda_2, \dots, \lambda_n) \leftrightarrow (x_1, x_2, \dots, x_n)$, $x_i \neq x_j$, $\lambda_i \neq \lambda_j$, that is, we have

$$\sin^-_\lambda(x) = \sin^-_x(\lambda), \qquad \cos^-_\lambda(x) = \cos^-_x(\lambda).$$

These relations expresses the *duality* of antisymmetric sine and cosine functions.

The duality is also true for symmetric trigonometric functions,

$$\sin^+_\lambda(x) = \sin^+_x(\lambda), \qquad \cos^+_\lambda(x) = \cos^+_x(\lambda).$$

Orthogonality on the fundamental domain $F(\tilde{S}^{\rm aff}_n)$. Antisymmetric multivariate sine functions $\sin^-_m$ with $m = (m_1, m_2, \dots, m_n) \in D^+_+$, $m_j \in \mathbb{Z}$, are orthogonal on $F(\tilde{S}^{\rm aff}_n)$ with respect to the Euclidean measure. We have

$$2^{2n} \int_{\overline{F(\tilde{S}^{\rm aff}_n)}} \sin^-_m(x) \sin^-_{m'}(x) dx = \delta_{m,m'}, \tag{15}$$

where the closure $\overline{F(\tilde{S}^{\rm aff}_n)}$ of $F(\tilde{S}^{\rm aff}_n)$ consists of points $x = (x_1, x_2, \dots, x_n) \in E_n$ such that

$$\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

This relation follows from orthogonality of the sine functions $\sin 2\pi m_i x_j$ (entering into the definition of the function $\sin^-_m(x)$). Indeed, we have

$$2^2 \int_0^{1/2} \sin(2\pi k t)\, \sin(2\pi k' t)\, dt = \delta_{kk'}.$$

Therefore, if T is the set $[0, \frac{1}{2}]^n$, then

$$2^{2n} \int_{\mathsf{T}} \sin^-_m(x) \sin^-_{m'}(x) dx = |S_n| \delta_{m,m'},$$

where $|S_n|$ is an order of the permutation group. Since $F(\tilde{S}^{\rm aff}_n)$ covers the set T exactly $|S_n|$ times, the formula (15) follows.

A similar orthogonality relation can be written down for the antisymmetric multivariate cosine functions,

$$2^{2n} \int_{\overline{F(\tilde{S}^{\rm aff}_n)}} \cos^-_m(x) \cos^-_{m'}(x)\, dx = \delta_{m,m'}. \tag{16}$$

For symmetric multivariate sine and cosine function we have the orthogonality relations

$$2^{2n} \int_{\overline{F(\tilde{S}^{\rm aff}_n)}} \sin^+_m(x) \sin^+_{m'}(x)\, dx = |G_m| \delta_{m,m'}, \tag{17}$$

$$2^{2n} \int_{\overline{F(\tilde{S}^{\rm aff}_n)}} \cos^+_m(x) \cos^+_{m'}(x)\, dx = |G_m| \delta_{m,m'}, \tag{18}$$

where m and m' are such that $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, $m'_1 \geq m'_2 \geq \cdots \geq m'_n \geq 0$, $m_i, m'_j \in \mathbb{Z}$, and $|G_m|$ is an order of the subgroup $G_m \subset S_n$ consisting of elements leaving m invariant.

Orthogonality of symmetric and antisymmetric trigonometric functions. Let w_i $(i = 1, 2, \ldots, n-1)$ be the permutation of coordinates x_i and x_{i+1}. We create the domain $F^{\rm ext}(\tilde{S}_n^{\rm aff}) = F(\tilde{S}_n^{\rm aff}) \cup w_i F(\tilde{S}_n^{\rm aff})$, where $F(\tilde{S}_n^{\rm aff})$ is the fundamental domain for the group $\tilde{S}_n^{\rm aff}$. Let $F^{\rm ext}$ be a closure of the domain $F^{\rm ext}(\tilde{S}_n^{\rm aff})$. If $i = 1$, then $F^{\rm ext}$ consists of points $x \in E_n$ such that

$$\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \quad \text{or} \quad \frac{1}{2} \geq x_2 \geq x_1 \geq x_3 \geq x_4 \geq \cdots \geq x_n \geq 0.$$

Since for $m = (m_1, m_2, \ldots, m_n) \in D_+ \equiv \overline{D_+^+}$, $m_j \in \mathbb{Z}$, we have

$$\cos^+_m(w_i x) = \cos^+_m(x), \qquad \sin^-_m(w_i x) = -\sin^-_m(x),$$

then

$$\int_{F^{\rm ext}} \sin^-_m(x) \cos^+_{m'}(x) dx = 0. \tag{19}$$

Indeed, due to symmetry and antisymmetry of symmetric and antisymmetric trigonometric functions, respectively, we have

$$\int_{F^{\rm ext}} \sin^-_m(x) \cos^+_{m'}(x) dx$$
$$= \int_{\overline{F(\tilde{S}_n^{\rm aff})}} \sin^-_m(x) \cos^+_{m'}(x) dx + \int_{\overline{w_i F(\tilde{S}_n^{\rm aff})}} \sin^-_m(x) \cos^+_{m'}(x) dx$$
$$= \int_{\overline{F(\tilde{S}_n^{\rm aff})}} \sin^-_m(x) \cos^+_{m'}(x) dx + \int_{\overline{F(\tilde{S}_n^{\rm aff})}} (-\sin^-_m(x)) \cos^+_{m'}(x) dx = 0.$$

For $n = 1$ the orthogonality (19) means the orthogonality of the functions sine and cosine on the interval $(0, 2\pi)$.

The relation

$$\int_{F^{\rm ext}} \sin^+_m(x) \cos^-_{m'}(x) dx = 0 \tag{20}$$

is proved similarly.

5. Special cases

For special values of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the function $\sin^-_\lambda(x)$ can be represented in a form of product of trigonometric functions of one variables. For $\lambda \equiv \rho_1 = (n, n-1, \ldots, 1)$ we get

$$\sin^-_{\rho_1}(x) = \prod_{1 \leq i < j \leq n} \sin\, \pi(x_i - x_j) \sin\, \pi(x_i + x_j) \prod_{1 \leq i \leq n} \sin 2\pi x_i. \tag{21}$$

In order to prove this formula we have to represent the sine functions of one variable in (3) and (21) in terms of exponential functions. Then we fulfil multiplications of all expressions in (21) and obtain $\sin^-_{\rho_1}(x)$ in the form of sum of products of exponential functions. Comparing this form with the expression (3) for $\sin^-_{\rho_1}(x)$ in terms of exponential functions we show that formula (21) is true.

For $\lambda \equiv \rho_2 = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2})$ we have

$$\sin^-_{\rho_2}(x) = \prod_{1 \leq i < j \leq n} \sin\, \pi(x_i - x_j) \sin\, \pi(x_i + x_j) \prod_{1 \leq i \leq n} \sin \pi x_i.$$

For $\lambda \equiv \rho_3 = (n-1, n-2, \dots, 1, 0)$ one has

$$\sin^-_{\rho_3}(x) = \prod_{1\le i<j\le n} \sin\, \pi(x_i - x_j) \sin\, \pi(x_i + x_j).$$

Similarly, for symmetric multivariate cosine functions we have

$$\cos^+_{\rho_1}(x) = \prod_{1\le i<j\le n} \cos\, \pi(x_i - x_j) \cos\, \pi(x_i + x_j) \prod_{1\le i\le n} \cos 2\pi x_i,$$

$$\cos^+_{\rho_2}(x) = \prod_{1\le i<j\le n} \cos\, \pi(x_i - x_j) \cos\, \pi(x_i + x_j) \prod_{1\le i\le n} \cos \pi x_i,$$

$$\cos^+_{\rho_3}(x) = \prod_{1\le i<j\le n} \cos\, \pi(x_i - x_j) \cos\, \pi(x_i + x_j).$$

These formulas are proved in the same way as the formula (21).

6. Solutions of the Laplace equation

The Laplace operator on the n-dimensional Euclidean space E_n in the orthogonal coordinates $x = (x_1, x_2, \dots, x_n)$ has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We take any summand in the expression for symmetric or antisymmetric multivariate sine or cosine function and act upon it by the operator Δ. We get

$$\Delta \sin 2\pi(w\lambda)_1 x_1 \sin 2\pi(w\lambda)_2 x_2 \cdots \sin 2\pi(w\lambda)_n x_n$$
$$= -4\pi^2\langle\lambda,\lambda\rangle \sin 2\pi(w\lambda)_1 x_1 \sin 2\pi(w\lambda)_2 x_2 \cdots \sin 2\pi(w\lambda)_n x_n,$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ determines $\sin^\pm_\lambda(x)$ or $\cos^\pm_\lambda(x)$ and $\langle\lambda,\lambda\rangle = \sum_{i=1}^n \lambda^2$. Since this action of Δ does not depend on a summand from the expression for symmetric or antisymmetric multivariate sine or cosine function, we have

$$\Delta \sin^\pm_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle \sin^\pm_\lambda(x), \tag{22}$$

$$\Delta \cos^\pm_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle \cos^\pm_\lambda(x). \tag{23}$$

The formulas (22) and (23) admit a generalization. Let $\sigma_k(y_1, y_2, \dots, y_n)$ be the k-th elementary symmetric polynomial of degree k of the variables $y_1, y_2, \dots, y_n$, that is,

$$\sigma_k(y_1, y_2, \dots, y_n) = \sum_{1\le k_1<k_2<\cdots<k_n\le n} y_{k_1} y_{k_2} \cdots y_{k_n}.$$

Then for $k = 1, 2, \dots, n$ we have

$$\sigma_k\left(\tfrac{\partial^2}{\partial x_1^2}, \tfrac{\partial^2}{\partial x_2^2}, \dots, \tfrac{\partial^2}{\partial x_n^2}\right) \sin^\pm_\lambda(x) = (-4\pi^2)^k \sigma_k(\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2) \sin^\pm_\lambda(x), \tag{24}$$

$$\sigma_k\left(\tfrac{\partial^2}{\partial x_1^2}, \tfrac{\partial^2}{\partial x_2^2}, \dots, \tfrac{\partial^2}{\partial x_n^2}\right) \cos^\pm_\lambda(x) = (-4\pi^2)^k \sigma_k(\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2) \cos^\pm_\lambda(x).$$

Note that n differential equations (24) are algebraically independent.

Thus, the functions $\sin^\pm_m(x)$, $\cos^\pm_m(x)$, $m = (m_1 m_2, \dots, m_n)$, $m_j \in \mathbb{Z}$, are eigenfunctions of the operators $\sigma_k\left(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \dots, \frac{\partial^2}{\partial x_n^2}\right)$, $k = 1, 2, \dots, n$, on the fundamental domain $F(\tilde{S}_n^{\rm aff})$ satisfying the boundary conditions formulated in section

4. For example, the functions $\sin^-_m(x)$ are eigenfunctions of these operators on the fundamental domain $F(\tilde{S}_n^{\rm aff})$ satisfying the boundary condition

$$\sin^-_m(x)=0 \qquad \text{for} \qquad x\in\partial F(\tilde{S}_n^{\rm aff}) \tag{25}$$

(the Dirichlet boundary value problem). The functions $\cos^+_m(x)$ are eigenfunctions of these operators on $F(\tilde{S}_n^{\rm aff})$ satisfying the boudary condition

$$\frac{\partial \cos^+_m(x)}{\partial \mathbf{n}}=0 \quad \text{for} \quad x\in\partial F(\tilde{S}_n^{\rm aff}),$$

that is, these functions give solutions of the Neumann boudary value problem on $\partial F(\tilde{S}_n^{\rm aff})$.

7. Symmetric and antisymmetric multivariate sine and cosine series

Symmetric and antisymmetric trigonometric functions determine symmetric and antisymmetric multivariate trigonometric Fourier transforms which generalize the usual trigonometric Fourier transforms.

As in the case of trigonometric functions of one variable, (anti)symmetric sine and cosine functions determine three types of trigonometric Fourier transforms:

(a) Fourier transforms related to the sine and cosine functions $\sin^\pm_m(x)$ and $\cos^\pm_m(x)$ with $m=(m_1,m_2,\dots,m_n)$, $m_j\in\mathbb{Z}$ (trigonometric Fourier series);

(b) Fourier transforms related to the sine and cosine functions $\sin^\pm_\lambda(x)$ and $\cos^\pm_\lambda(x)$ with $\lambda\in D_+\equiv\overline{D_+^+}$ (integral Fourier transforms);

(c) Symmetric and antisymmetric multivariate finite sine and cosine Fourier transforms.

In this section we consider the case (a). Let $f(x)$ be an antisymmetric (with respect to the extended affine symmetric group $\tilde{S}_n^{\rm aff}$) continuous real function on the n-dimensional Euclidean space E_n, which has continuous derivatives and vanishes on the boundary $\partial F(\tilde{S}_n^{\rm aff})$ of the fundamental domain $F(\tilde{S}_n^{\rm aff})$, that is, $f(x)$ behaves under action of elements of $\tilde{S}_n^{\rm aff}$ as the functions $\sin^-_m(x)$ do. We may consider this function on the set $\mathsf{T}=[0,\frac12]^n$ (this set is a closure of the union of the sets $wF(\tilde{S}_n^{\rm aff})$, $w\in S_n$). Then $f(x)$, as a function on T, can be expanded in sine functions

$$\sin 2\pi m_1x_1\cdot\sin 2\pi m_2x_2\cdots\sin 2\pi m_nx_n, \quad m_i\in\mathbb{Z}^{>0}.$$

We have

$$f(x)=\sum_{m_i\in\mathbb{Z}^{>0}} c_m\,\sin 2\pi m_1x_1\cdot\sin 2\pi m_2x_2\cdots\sin 2\pi m_nx_n, \tag{26}$$

where $m=(m_1,m_2,\dots,m_n)$. Let us show that $c_{wm}=(\det w)c_m$, $w\in S_n$. We represent each sine function in the expression (26) in the form $\sin\alpha=(2{\rm i})^{-1}(e^{{\rm i}\alpha}-e^{-{\rm i}\alpha})$. Then

$$f(x)=\sum_{m_i\in\mathbb{Z}} c_m e^{2\pi{\rm i}m_1x_1}e^{2\pi{\rm i}m_2x_2}\cdots e^{2\pi{\rm i}m_nx_n}=\sum_{m_i\in\mathbb{Z}} c_m e^{2\pi{\rm i}\langle m,x\rangle},$$

where $\langle m,x\rangle=\sum_{i=1}^n m_ix_i$ and c_m with positive m_i, $i=1,2,\dots,n$, are such as in (26) and each change of a sign in m leads to multiplication of c_m by (-1). Due to

the property $f(wx) = (\det w)f(x)$, $w \in S_n$, for any $w \in S_n$ we have

$$f(wx) = \sum_{m_i\in\mathbb{Z}} c_m e^{2\pi \mathrm{i} m_1 x_{w(1)}} \cdots e^{2\pi \mathrm{i} m_n x_{w(n)}} = \sum_{m_i\in\mathbb{Z}} c_m e^{2\pi \mathrm{i} m_{w^{-1}(1)} x_1} \cdots e^{2\pi \mathrm{i} m_{w^{-1}(n)} x_n}$$
$$= \sum_{m_i\in\mathbb{Z}} c_{wm} e^{2\pi \mathrm{i} m_1 x_1} \cdots e^{2\pi \mathrm{i} m_n x_n} = (\det w) f(x)$$
$$= (\det w) \sum_{m_i\in\mathbb{Z}} c_m e^{2\pi \mathrm{i} m_1 x_1} \cdots e^{2\pi \mathrm{i} m_n x_n}.$$

Thus, the coefficients c_m in (26) satisfy the conditions $c_{wm} = (\det w)c_m$, $w \in S_n$.

Collecting in (26) products of sine functions at $(\det w)c_{wm}$, $w \in S_n$, we obtain the expansion

$$f(x) = \sum_{m\in P_+^+} c_m \det\left(\sin 2\pi m_i x_j\right)_{i,j=1}^n \equiv \sum_{m\in P_+^+} c_m \sin^-_m(x), \tag{27}$$

where $P_+^+ := D_+^+ \cap \mathbb{Z}^n$. Thus, *any antisymmetric (with respect to S_n) continuous real function f on T, which has continuous derivatives, can be expanded in antisymmetric multivariate sine functions* $\sin^-_m(x)$, $m \in P_+^+$.

By the orthogonality relation (15), the coefficients c_m in the expansion (27) are determined by the formula

$$c_m = 2^{2n} \int_{F(S_n^{\mathrm{aff}})} f(x) \det\left(\sin 2\pi m_i x_j\right)_{i,j=1}^n dx = 2^{2n} \int_{F(S_n^{\mathrm{aff}})} f(x) \sin^-_m(x) dx, \tag{28}$$

Moreover, the Plancherel formula

$$\sum_{m\in P_+^+} |c_m|^2 = 2^{2n} \int_{F(S_n^{\mathrm{aff}})} |f(x)|^2 dx$$

holds, which means that the Hilbert spaces with the appropriate scalar products are isometric.

Formula (28) is an antisymmetrized sine Fourier transform of the function $f(x)$. Formula (27) gives an inverse transform. Formulas (27) and (28) give the *antisymmetric multivariate sine Fourier transforms* corresponding to antisymmetric sine functions $\sin^-_m(x)$, $m \in P_+^+$.

Analogous transforms hold for symmetric cosine functions $\cos^+_m(x)$, $m \in P_+ = D_+ \cap \mathbb{Z}^n$. Let $f(x)$ be a symmetric (with respect to the group $\tilde{S}_n^{\mathrm{aff}}$) continuous real function on the n-dimensional Euclidean space E_n, which has continuous derivatives, that is, $f(x)$ behaves under action of elements of $\tilde{S}_n^{\mathrm{aff}}$ as the functions $\cos^+_m(x)$ do. We may consider this function as a function on $F(\tilde{S}_n^{\mathrm{aff}})$. Then we can expand this function as

$$f(x) = \sum_{m\in P_+} c_m {\det}^+\left(\cos 2\pi m_i x_j\right)_{i,j=1}^n = \sum_{m\in P_+} c_m \cos^+_m(x), \tag{29}$$

where $m = (m_1, m_2, \ldots, m_n)$ are *integer* n-tuples such that $m_1 \ge m_2 \ge \cdots \ge m_n \ge 0$. The coefficients c_m of this expansion are given by the formula

$$c_m = 2^{2n} |G_m|^{-1} \int_{F(S_n^{\mathrm{aff}})} f(x) \cos^+_m(x) dx. \tag{30}$$

The Plancherel formula is of the form

$$\sum_{m\in P_+} |G_m||c_m|^2 = 2^{2n} \int_{F(S_n^{\rm aff})} |f(x)|^2 dx.$$

Now let $f(x)$ be an antisymmetric (with respect to the permutation group S_n) continuous real function on the set $\mathsf{T} = [0,\frac12]^n$, which has continuous derivatives and vanishes on the boundary $\partial F(\tilde S_n^{\rm aff})$ of the fundamental domain $F(\tilde S_n^{\rm aff})$. Then $f(wx) = (\det w) f(x)$, $w \in S_n$. We consider this function as a function on $F(\tilde S_n^{\rm aff})$. One has the expansion

$$f(x) = \sum_{m\in P_+^+} c_m \det\left(\cos 2\pi m_i x_j\right)_{i,j=1}^n \equiv \sum_{m\in P_+^+} c_m {\rm cos}_m^-(x), \tag{31}$$

where

$$c_m = 2^{2n} \int_{F(S_n^{\rm aff})} f(x) {\rm cos}_m^-(x) dx. \tag{32}$$

Moreover, the Plancherel formula $\sum_{m\in P_+^+} |c_m|^2 = 2^{2n} \int_{F(S_n^{\rm aff})} |f(x)|^2 dx$ holds.

A similar expansion for the functions ${\rm sin}_m^+(x)$, $m \in P_+$, is of the form

$$f(x) = \sum_{m\in P_+} c_m {\rm det}^+ (\sin 2\pi m_i x_j)_{i,j=1}^n \equiv \sum_{m\in P_+} c_m {\rm sin}_m^+(x), \tag{33}$$

where the coefficients c_m are given by

$$c_m = 2^{2n} |G_m|^{-1} \int_{F(S_n^{\rm aff})} f(x) {\rm sin}_m^+(x) dx. \tag{34}$$

The Plancherel formula is of the form $\sum_{m\in P_+} |G_m||c_m|^2 = 2^{2n} \int_{F(S_n^{\rm aff})} |f(x)|^2 dx$.

8. Symmetric and antisymmetric multivariate sine and cosine Fourier transforms on $F(\tilde S_n)$

The expansions of the previous subsection give expansions of functions on the fundamental domain $F(\tilde S_n^{\rm aff})$ in functions ${\rm sin}_m^\pm(x)$ and ${\rm cos}_m^\pm(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$. The functions ${\rm sin}_\lambda^\pm(x)$ and ${\rm cos}_\lambda^\pm(x)$ with λ lying in the fundamental domain $F(\tilde S_n)$ (and not obligatory integral) are invariant (anti-invariant) only with respect to the extended permutation group $\tilde S_n$. A fundamental domain of $\tilde S_n$ coincides with the set D_+^+ from section 3. The sine and cosine functions ${\rm sin}_\lambda^\pm(x)$ and ${\rm cos}_\lambda^\pm(x)$, determined by $\lambda \in D_+$, give Fourier transforms on the domain D_+.

We began with the usual sine Fourier transforms on $\mathbb{R}_+^n$:

$$\tilde f(\lambda) = \int_{\mathbb{R}_+^n} f(x) \sin 2\pi\lambda_1 x_1 \ \sin 2\pi\lambda_2 x_2 \cdots \sin 2\pi\lambda_n x_n \, dx, \tag{35}$$

$$f(x) = 2^{2n} \int_{\mathbb{R}_+^n} \tilde f(\lambda) \sin 2\pi\lambda_1 x_1 \ \sin 2\pi\lambda_2 x_2 \cdots \sin 2\pi\lambda_n x_n \, d\lambda. \tag{36}$$

Let the function $f(x)$, given on $\mathbb{R}_+^n$, be anti-invariant with respect to the symmetric group S_n, that is, $f(wx) = (\det w) f(x)$, $w \in S_n$. It is easy to check that the function $\tilde f(\lambda)$ is also anti-invariant with respect to the group S_n. Replace in (35) λ by $w\lambda$,

$w \in S_n$, multiply both sides by $\det w$, and sum these both side over $w \in S_n$. Due to the expression (3) for symmetric sine functions $\sin^-_\lambda(x)$, instead of (35) we obtain

$$\tilde{f}(\lambda) = |S_n|^{-1} \int_{\mathbb{R}^n_+} f(x) \sin^-_\lambda(x) dx \equiv \int_{D_+} f(x) \sin^-_\lambda(x) dx, \qquad \lambda \in D^+_+, \tag{37}$$

where we have taken into account that $f(x)$ is anti-invariant with respect to S_n.

Starting from (36), we obtain the inverse formula,

$$f(x) = 2^{2n} \int_{D_+} \tilde{f}(\lambda) \sin^-_\lambda(x) d\lambda. \tag{38}$$

For the transforms (37) and (38) the Plancherel formula

$$\int_{D_+} |f(x)|^2 dx = 2^{2n} \int_{D_+} |\tilde{f}(\lambda)|^2 d\lambda$$

holds. The formulas (37) and (38) determine the *antisymmetric multivariate sine Fourier transforms on the domain* $F(\tilde{S}_n)$.

Similarly, starting from formulas (35) and (36) we receive the symmetric multivariate sine Fourier transforms on the domain $F(\tilde{S}_n)$:

$$\tilde{f}(\lambda) = \int_{D_+} f(x) \sin^+_\lambda(x) dx, \qquad \lambda \in D_+, \tag{39}$$

$$f(x) = 2^{2n} \int_{D_+} \tilde{f}(\lambda) \sin^+_\lambda(x) d\lambda. \tag{40}$$

The corresponding Plancherel formula holds.

The cosine functions $\cos^\pm_\lambda(x)$ determine similar transforms. Namely, we have

$$\tilde{f}(\lambda) = \int_{D_+} f(x) \det\left(\cos 2\pi\lambda_i x_j\right)^n_{i,j=1} dx \equiv \int_{D_+} f(x) \cos^-_\lambda(x) dx, \tag{41}$$

where

$$f(x) = 2^{2n} \int_{D_+} \tilde{f}(\lambda) \det\left(\cos 2\pi\lambda_i x_j\right)^n_{i,j=1} d\lambda \equiv 2^{2n} \int_{D_+} \tilde{f}(\lambda) \cos^-_\lambda(x) d\lambda, \tag{42}$$

and

$$\tilde{f}(\lambda) = \int_{D_+} f(x) \det{}^+\left(\cos 2\pi\lambda_i x_j\right)^n_{i,j=1} dx \equiv \int_{D_+} f(x) \cos^+_\lambda(x) dx, \tag{43}$$

where

$$f(x) = 2^{2n} \int_{D_+} \tilde{f}(\lambda) \det{}^+\left(\cos 2\pi\lambda_i x_j\right)^n_{i,j=1} d\lambda \equiv 2^{2n} \int_{D_+} \tilde{f}(\lambda) \cos^+_\lambda(x) d\lambda. \tag{44}$$

The corresponding Plancherel formulas hold.

9. Finite 1-dimensional sine and cosine transforms

Finite one-dimensional sine and cosine transforms are useful for applications. The theory of these transforms as well as their different applications and methods of work with them are given in Ref. 17. In this section we give these one-dimensional transforms in the form[11] which will be used in the following sections.

Let N be a positive integer. To this number there corresponds the finite set of points (the grid) $\frac{r}{N}$, $r = 0, 1, 2, \ldots, N$. We denote this set by F_N,

$$F_N = \left\{0, \tfrac{1}{N}, \tfrac{2}{N}, \ldots, \tfrac{N-1}{N}, 1\right\}. \tag{45}$$

We consider sine functions on the grid F_N, that is, the functions

$$\varphi_m(s) := \sin(\pi m s), \quad s \in F_N, \quad m \in \mathbb{Z}^{\geq}. \tag{46}$$

Since $\varphi_m(s) = \pm\varphi_{m+N}(s)$ and $\varphi_0(s) = \varphi_N(s) = 0$, we consider these discrete functions only for

$$m \in D_N := \{1, 2, \dots, N-1\}.$$

The functions (46) vanish on the points 0 and 1 of F_N. For this reason, we also consider the subset

$$F_N^- = \left\{\tfrac{1}{N}, \tfrac{2}{N}, \dots, \tfrac{N-1}{N}\right\} \quad (N-1 \text{ points})$$

of the grid F_N.

The functions (46) are orthogonal on F_N^- and the orthogonality relation is of the form

$$\langle \varphi_m, \varphi_{m'} \rangle = \sum_{s \in F_N^-} \varphi_m(s)\varphi_{m'}(s) = \frac{N}{2}\delta_{mm'}, \qquad m, m' \in D_N. \tag{47}$$

They determine the following expansion of functions, given on the grid F_N^-:

$$f(s) = \sum_{m=1}^{N-1} a_m \sin(\pi m s), \tag{48}$$

where the coefficients a_m are given by

$$a_m = \frac{2}{N} \sum_{s \in F_N^-} f(s) \sin(\pi m s). \tag{49}$$

Formulas (48) and (49) determine the *discrete sine transform.*

We also consider cosine functions on the grid F_N, that is, the functions

$$\phi_m(s) = \cos(\pi m s), \quad s \in F_N, \quad m \in \{0, 1, 2, \dots N\}. \tag{50}$$

These functions are orthogonal on the grid F_N with the orthogonality relation

$$\langle \phi_m, \phi_{m'} \rangle = \sum_{s \in F_N} c_s \phi_m(s) \phi_{m'}(s) = r_m N \delta_{mm'}, \tag{51}$$

where $r_m = 1$ for $m = 0, N$ and $r_m = \frac{1}{2}$ otherwise, $c_s = \frac{1}{2}$ for $s = 0$, 1 and $c_s = 1$ otherwise[18].

The functions (50) determine an expansion of functions on the grid F_N as

$$f(s) = \sum_{m=0}^{N} b_m \cos(\pi m s), \quad s \in F_N, \tag{52}$$

where the coefficients b_m are given by

$$b_m = r_m^{-1} N^{-1} \sum_{s \in F_N} c_s f(s) \cos(\pi m s). \tag{53}$$

Formulas (52) and (53) determine the *discrete cosine transform.*

10. Antisymmetric multivariate finite sine transforms

The finite sine and cosine transforms of the previous section can be generalized to the n-dimensional case in symmetric and antisymmetric forms. In fact, these generalizations are finite (anti)symmetric multivariate trigonometric transforms. They are derived by using 1-dimensional finite sine and cosine transforms. Some of the transforms can be also derived by using the results of Ref. 18. In order to introduce multivariate finite sine transforms we have to define (anti)symmetric multivariate finite sine functions. Note that notations $\sin^{\pm}_{\mathbf{m}}(\mathbf{s})$ in this section slightly differ from notations of section 2.

We take the discrete sine function (46) and make a multivariate discrete sine function by multiplying n copies of this function:

$$\sin_{\mathbf{m}}(\mathbf{s}) := \sin(\pi m_1 s_1)\sin(\pi m_2 s_2)\cdots\sin(\pi m_n s_n), \tag{54}$$
$$s_j \in F_N, \qquad m_i \in D_N \equiv \{1,2,\dots,N-1\},$$

where $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{m} = (m_1, m_2, \dots, m_n)$. We take these multivariate functions for integers m_i such that $N > m_1 > m_2 > \cdots > m_n > 0$ and make an antisymmetrization to obtain a finite version of the antisymmetric multivariate sine function (3):

$$\sin^-_{\mathbf{m}}(\mathbf{s}) := |S_n|^{-1/2}\det(\sin \pi m_i s_j)^n_{i,j=1}, \tag{55}$$

where $|S_n|$ is an order of the symmetric group S_n. (We have here expressions $\sin \pi m_i s_j$, not $\sin 2\pi m_i s_j$ as in (3).) Since functions $\sin \pi m_i s_j$ are considered for positive m_i and s_j, we deal here with the permutation group S_n instead of the group $\tilde{S}_n$.

The n-tuple $\mathbf{s}$ in (55) runs over $(F^-_N)^n \equiv F^-_N \times \cdots \times F^-_N$ (n times). We denote by $\hat{F}^n_N$ the subset of $(F^-_N)^n$ consisting of $\mathbf{s} \in (F^-_N)^n$ such that

$$s_1 > s_2 > \cdots > s_n.$$

Note that s_i may take the values $\frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}$. Acting by permutations $w \in S_n$ upon $\hat{F}^n_N$ we obtain the whole set $(F^-_N)^n$ without those points which are invariant under some nontrivial permutation $w \in S_n$. Due to antisymmetricity, the functions (55) vanishes on the last points.

We denote by $\hat{D}^n_N$ the set of integer n-tuples $\mathbf{m} = (m_1, m_2, \dots, m_n)$ such that

$$N > m_1 > m_2 > \cdots > m_n > 0.$$

We need a scalar product of functions (55). For this we define a scalar product of functions (54) as

$$\langle \sin_{\mathbf{m}}(\mathbf{s}), \sin_{\mathbf{m}'}(\mathbf{s})\rangle = \prod_{i=1}^n \langle \varphi_{m_i}(s_i), \varphi_{m'_i}(s_i)\rangle = \left(\frac{N}{2}\right)^n \delta_{\mathbf{m},\mathbf{m}'},$$

where the scalar product $\langle \varphi_{m_i}(s_i), \varphi_{m'_i}(s_i)\rangle$ is given by formula (47). Since functions $\sin^-_{\mathbf{m}}(\mathbf{s})$ are linear combinations of functions $\sin_{\mathbf{m}'}(\mathbf{s})$, a scalar product for $\sin^-_{\mathbf{m}}(\mathbf{s})$ is also defined.

Proposition 1. *For* $\mathbf{m}, \mathbf{m}' \in \hat{D}^n_N$, *the discrete functions* (55) *satisfy the orthogonality relation*

$$\langle \sin^-_{\mathbf{m}}(\mathbf{s}), \sin^-_{\mathbf{m}'}(\mathbf{s})\rangle = \sum_{\mathbf{s}\in(F^-_N)^n} \sin^-_{\mathbf{m}}(\mathbf{s})\sin^-_{\mathbf{m}'}(\mathbf{s}) = |S_n| \sum_{\mathbf{s}\in\hat{F}^n_N} \sin^-_{\mathbf{m}}(\mathbf{s})\sin^-_{\mathbf{m}'}(\mathbf{s})$$

$$= (N/2)^n \delta_{\mathbf{m}\mathbf{m}'}. \tag{56}$$

Proof. Since $N > m_1 > m_2 > \cdots > m_n > 0$ for $\mathbf{m} \in \hat{D}_N^n$, then due to the orthogonality relation (47) for the sine functions $\sin(\pi m s)$ we have

$$\sum_{\mathbf{s}\in(F_N^-)^n} \mathrm{sin}^-_{\mathbf{m}}(\mathbf{s})\mathrm{sin}^-_{\mathbf{m}'}(\mathbf{s}) = |S_n|^{-1} \sum_{w\in S_n} \prod_{i=1}^{n} \sum_{s_i=1}^{N-1} \sin(\pi m_{w(i)} s_i) \sin(\pi m'_{w(i)} s_i)$$

$$= (N/2)^n \delta_{\mathbf{m}\mathbf{m}'},$$

where $(m_{w(1)}, m_{w(2)}, \dots, m_{w(n)})$ is obtained from $(m_1, m_2, \dots, m_n)$ by action by the permutation $w \in S_n$. Since functions $\mathrm{sin}^-_{\mathbf{m}}(\mathbf{s})$ are antisymmetric with respect to S_n, we have

$$\sum_{\mathbf{s}\in(F_N^-)^n} \mathrm{sin}^-_{\mathbf{m}}(\mathbf{s})\mathrm{sin}^-_{\mathbf{m}'}(\mathbf{s}) = |S_n| \sum_{\mathbf{s}\in\hat{F}_N^n} \mathrm{sin}^-_{\mathbf{m}}(\mathbf{s})\mathrm{sin}^-_{\mathbf{m}'}(\mathbf{s}).$$

This proves the proposition.

Let f be a function on $\hat{F}_N^n$ (or an antisymmetric function on $(F_N^-)^n$). Then it can be expanded in functions (55) as

$$f(\mathbf{s}) = \sum_{\mathbf{m}\in\hat{D}_N^n} a_{\mathbf{m}} \mathrm{sin}^-_{\mathbf{m}}(\mathbf{s}), \tag{57}$$

where the coefficients $a_{\mathbf{m}}$ are determined by the formula

$$a_{\mathbf{m}} = (2/N)^n |S_n| \sum_{\mathbf{s}\in\hat{F}_N^n} f(\mathbf{s}) \mathrm{sin}^-_{\mathbf{m}}(\mathbf{s}). \tag{58}$$

A validity of the expansions (57) and (58) follows from the facts that numbers of elements in $\hat{D}_N^n$ and in $\hat{F}_N^n$ are the same and from the orthogonality relation (56).

11. Symmetric multivariate finite cosine transforms

We take the finite cosine functions (50) and make multivariate finite cosine functions by multiplying n copies of this function:

$$\cos_{\mathbf{m}}(\mathbf{s}) := \cos(\pi m_1 s_1)\cos(\pi m_2 s_2)\cdots\cos(\pi m_n s_n), \tag{59}$$

$$s_j \in F_N, \qquad m_i \in \{0, 1, 2, \dots, N\}.$$

We consider these functions for integers m_i such that $N \ge m_1 \ge m_2 \ge \cdots \ge m_n \ge 0$ (the collection of these n-tuples $\mathbf{m} = (m_1, m_2, \dots, m_n)$ will be denoted by $\breve{D}_N^n$) and make a symmetrization. As a result, we obtain a finite version of the symmetric multivariate cosine function (6):

$$\cos^+_{\mathbf{m}}(\mathbf{s}) := |S_n|^{-1/2} \sum_{w\in S_n} \cos(\pi m_{w(1)} s_1)\cos(\pi m_{w(2)} s_2)\cdots\cos(\pi m_{w(n)} s_n). \tag{60}$$

(We have here expressions $\cos \pi m_i s_j$, not $\cos 2\pi m_i s_j$ as in (6). Therefore, the notation $\cos^+_{\mathbf{m}}(\mathbf{s})$ here slightly differs from the notation in section 2.)

The n-tuple $\mathbf{s}$ in (60) runs over F_N^n. We denote by $\breve{F}_N^n$ the subset of F_N^n consisting of $\mathbf{s} \in F_N^n$ such that

$$s_1 \ge s_2 \ge \cdots \ge s_n.$$

Note that s_i here may take the values $0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1$. Acting by permutations $w \in S_n$ upon $\breve{F}_N^n$ we obtain the whole set F_N^n, where points, invariant under some nontrivial permutation $w \in S_n$, are repeated several times. It is easy to see that a

point $\mathbf{s}_0 \in F_N^n$ is repeated $|S_{\mathbf{s}_0}|$ times in the set $\{w\breve{F}_N^n;\ w \in S_n\}$, where $|S_{\mathbf{s}_0}|$ is an order of the subgroup $S_{\mathbf{s}_0} \subset S_n$, whose elements leave $\mathbf{s}_0$ invariant.

A scalar product of functions (59) is determined by

$$\langle \mathrm{cos}_{\mathbf{m}}(\mathbf{s}), \mathrm{cos}_{\mathbf{m}'}(\mathbf{s})\rangle = \prod_{i=1}^{n} \langle \mathrm{cos}_{m_i}(s_i), \mathrm{cos}_{m_i'}(s_i)\rangle = N^n r_{m_1} \cdots r_{m_n} \delta_{\mathbf{m},\mathbf{m}'},$$

where the scalar product $\langle \mathrm{cos}_{m_i}(s_i), \mathrm{cos}_{m_i'}(s_i)\rangle$ is given by (51). Since functions $\mathrm{cos}^+_{\mathbf{m}}(\mathbf{s})$ are linear combinations of functions $\mathrm{cos}_{\mathbf{m}'}(\mathbf{s})$, then a scalar product for $\mathrm{cos}^+_{\mathbf{m}}(\mathbf{s})$ is also defined.

Proposition 2. *For $\mathbf{m}, \mathbf{m}' \in \breve{D}_N^n$, the discrete functions* (60) *satisfy the orthogonality relation*

$$\begin{aligned}\langle \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}), \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s})\rangle &= \sum_{\mathbf{s}\in F_N^n} c_{\mathbf{s}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s}) \\ &= |S_n| \sum_{\mathbf{s}\in \breve{F}_N^n} |S_{\mathbf{s}}|^{-1} c_{\mathbf{s}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s}) \\ &= N^n r_{\mathbf{m}} |S_{\mathbf{m}}| \delta_{\mathbf{m}\mathbf{m}'},\end{aligned} \tag{61}$$

where $c_{\mathbf{s}} = c_{s_1} c_{s_2} \cdots c_{s_n}$, $r_{\mathbf{s}} = r_{m_1} r_{m_2} \cdots r_{m_n}$, and c_{s_i} and r_{m_i} are such as in formula (51).

Proof. Due to the orthogonality relation for the cosine functions $\phi_m(s) = \cos(\pi m s)$ (see formula (51)) we have

$$\begin{aligned}\sum_{\mathbf{s}\in F_N^n} c_{\mathbf{s}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s}) &= \frac{|S_{\mathbf{m}}|}{|S_n|^{-1}|} \sum_{w\in S_n} \prod_{i=1}^{n} \sum_{s_i=0}^{N} c_{s_i} \cos(\pi m_{w(i)} s_i) \cos(\pi m'_{w(i)} s_i) \\ &= |S_{\mathbf{m}}| N^n r_{\mathbf{m}} \delta_{\mathbf{m}\mathbf{m}'},\end{aligned} \tag{62}$$

where $(m_{w(1)}, m_{w(2)}, \ldots, m_{w(n)})$ is obtained from $(m_1, m_2, \ldots, m_n)$ by action by the permutation $w \in S_n$. Since functions $\mathrm{cos}^+_{\mathbf{m}}(\mathbf{s})$ are symmetric with respect to S_n, we have

$$\sum_{\mathbf{s}\in F_N^n} c_{\mathbf{s}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s}) = |S_n| \sum_{\mathbf{s}\in \breve{F}_N^n} |S_{\mathbf{s}}|^{-1} c_{\mathbf{s}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}'}(\mathbf{s}).$$

This proves the proposition.

Let f be a function on $\breve{F}_N^n$ (or a symmetric function on F_N^n). Then it can be expanded in functions (60) as

$$f(\mathbf{s}) = \sum_{\mathbf{m}\in \breve{D}_N^n} a_{\mathbf{m}} \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}), \tag{63}$$

where the coefficients $a_{\mathbf{m}}$ are determined by the formula

$$\begin{aligned}a_{\mathbf{m}} &= N^{-n} |S_{\mathbf{m}}|^{-1} r_{\mathbf{m}}^{-1} \langle f(\mathbf{s}), \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s})\rangle \\ &= N^{-n} |S_{\mathbf{m}}|^{-1} r_{\mathbf{m}}^{-1} |S_n| \sum_{\mathbf{s}\in \breve{F}_N^n} |S_{\mathbf{s}}|^{-1} c_{\mathbf{s}} f(\mathbf{s}) \mathrm{cos}^+_{\mathbf{m}}(\mathbf{s}).\end{aligned} \tag{64}$$

A validity of the expansions (63) and (64) follows from the fact that numbers of elements in $\breve{D}_N^n$ and $\breve{F}_N^n$ are the same and from the orthogonality relation (61).

12. Other 1-dimensional finite cosine transforms

Along with the finite cosine transform of section 9 there exist other 1-dimensional finite transforms with the discrete cosine function as a kernel (see, for example, Refs. 19 and 20). In Ref. 19 the finite cosine transforms are denoted as DCT-1, DCT-2, DCT-3, DCT-4. The transform DCT-1 is in fact the transform, considered in section 9. Let us expose all these transforms (including the transform DCT-1), conserving notations used in the literature on signal processing. They are determined by a positive integer N.

DCT-1. This transform is given by the kernel

$$\mu_r(k) = \cos\frac{\pi rk}{N}, \qquad \text{where} \qquad k, r \in \{0, 1, 2, \ldots, N\}.$$

The orthogonality relation for these discrete functions is given by

$$\sum_{k=0}^{N} c_k \cos\frac{\pi rk}{N} \cos\frac{\pi r'k}{N} = h_r \frac{N}{2}\delta_{rr'}, \tag{65}$$

where $c_k = \frac{1}{2}$ for $k = 0, N$ and $c_k = 1$ otherwise, $h_r = 2$ for $r = 0, N$ and $h_r = 1$ otherwise.

Thus, these functions give the expansion

$$f(k) = \sum_{r=0}^{N} a_r \cos\frac{\pi rk}{N}, \qquad \text{where} \qquad a_r = \frac{2}{h_r N}\sum_{k=0}^{N} c_k f(k) \cos\frac{\pi rk}{N}. \tag{66}$$

DCT-2. This transform is given by the kernel

$$\omega_r(k) = \cos\frac{\pi(r+\frac{1}{2})k}{N}, \qquad \text{where} \qquad k, r \in \{0, 1, 2, \ldots, N-1\}.$$

The orthogonality relation for these discrete functions is given by

$$\sum_{k=0}^{N-1} c_k \cos\frac{\pi(r+\frac{1}{2})k}{N} \cos\frac{\pi(r'+\frac{1}{2})k}{N} = \frac{N}{2}\delta_{rr'}, \tag{67}$$

where $c_k = 1/2$ for $k = 0$ and $c_k = 1$ otherwise.

These functions determine the expansion

$$f(k) = \sum_{r=0}^{N-1} a_r \omega_r(k), \qquad \text{where} \qquad a_r = \frac{2}{N}\sum_{k=0}^{N-1} c_k f(k)\omega_r(k). \tag{68}$$

DCT-3. This transform is determined by the kernel

$$\sigma_r(k) = \cos\frac{\pi r(k+\frac{1}{2})}{N},$$

where k and r run over the values $\{0, 1, 2, \ldots, N-1\}$. The orthogonality relation for these discrete functions is given by the formula

$$\sum_{k=0}^{N-1} \cos\frac{\pi r(k+\frac{1}{2})}{N} \cos\frac{\pi r'(k+\frac{1}{2})}{N} = h_r \frac{N}{2}\delta_{rr'}, \tag{69}$$

where $h_k = 2$ for $k = 0$ and $h_k = 1$ otherwise.

These functions give the expansion

$$f(k)=\sum_{r=0}^{N-1} a_r \cos\frac{\pi r(k+\frac12)}{N}, \quad \text{where} \quad a_r=\frac{2}{h_r N}\sum_{k=0}^{N-1} f(k)\cos\frac{\pi r(k+\frac12)}{N}. \tag{70}$$

DCT-4. This transform is given by the kernel

$$\tau_r(k)=\cos\frac{\pi(r+\frac12)(k+\frac12)}{N},$$

where k and r run over the values $\{0,1,2,\dots,N-1\}$. The orthogonality relation for these discrete functions is given by

$$\sum_{k=0}^{N-1}\cos\frac{\pi(r+\frac12)(k+\frac12)}{N}\cos\frac{\pi(r'+\frac12)(k+\frac12)}{N}=\frac{N}{2}\delta_{rr'}. \tag{71}$$

These functions determine the expansion

$$f(k)=\sum_{r=0}^{N-1} a_r\cos\frac{\pi(r+\frac12)(k+\frac12)}{N}, \ \text{ where } \ a_r=\frac{2}{N}\sum_{k=0}^{N-1} f(k)\cos\frac{\pi(r+\frac12)(k+\frac12)}{N}. \tag{72}$$

Note that there exist also four discrete sine transforms, corresponding to the above discrete cosine transforms. They are obtained from the cosine transforms by replacing in (66), (68), (70) and (72) cosines discrete functions by sine discrete functions[17,20].

13. Other antisymmetric multivariate finite cosine transforms

Each of the finite cosine transforms DCT-1, DCT-2, DCT-3, DCT-4 generates the corresponding antisymmetric multivariate finite cosine transform. We call them AMDCT-1, AMDCT-2, AMDCT-3 and AMDCT-4. Let us give these transforms without proof. Their proofs are the same as in the case of symmetric multivariate finite cosine transforms of section 11. Below we use the notation $\tilde D_N^n$ for the subset of the set $D_N^n \equiv D_N\times D_N\times\cdots\times D_N$ (n times) with $D_N=\{0,1,2,\dots,N\}$ consisting of points $\mathbf{r}=(r_1,r_2,\dots,r_n)$, $r_i\in D_N$, such that

$$N\ge r_1>r_2>\cdots>r_n\ge 0.$$

AMDCT-1. This transform is given by the kernel

$$\cos^-_{\mathbf r}(\mathbf k)\equiv \cos^{1,-}_{\mathbf r}(\mathbf k)=|S_n|^{-1/2}\det\left(\cos\frac{\pi r_i k_j}{N}\right)_{i,j=1}^{n}, \tag{73}$$

where $\mathbf r\in\tilde D_N^n$ and $\mathbf k=(k_1,k_2,\dots,k_n)$, $k_i\in\{0,1,2,\dots,N\}$. The orthogonality relation for these kernels is

$$\langle \cos^-_{\mathbf r}(\mathbf k),\cos^-_{\mathbf r'}(\mathbf k)\rangle=|S_n|\sum_{\mathbf k\in\tilde D_N^n} c_{\mathbf k}\cos^-_{\mathbf r}(\mathbf k)\cos^-_{\mathbf r'}(\mathbf k)=h_{\mathbf r}\left(\frac N2\right)^n\delta_{\mathbf{rr}'}, \tag{74}$$

where

$$c_{\mathbf k}=c_1c_2\cdots c_n,\quad h_{\mathbf k}=h_1h_2\cdots h_n,$$

and c_i and h_j are such as in formula (65).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in\tilde{D}_N^n} a_{\mathbf{r}} \cos^-_{\mathbf{r}}(\mathbf{k}), \quad \text{where} \quad a_{\mathbf{r}} = h_{\mathbf{r}}^{-1}|S_n| \left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in\tilde{D}_N^n} c_{\mathbf{k}} f(\mathbf{k}) \cos^-_{\mathbf{r}}(\mathbf{k}). \quad (75)$$

The corresponding Plancherel formula is

$$|S_n| \sum_{\mathbf{k}\in\tilde{D}_N^n} c_{\mathbf{k}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in\tilde{D}_N^n} h_{\mathbf{r}} |a_{\mathbf{r}}|^2.$$

AMDCT-2. Let $\tilde{D}_{N-1}^n$ be the subset of D_{N-1}^n (with $D_{N-1} = \{0, 1, \dots, N-1\}$) consisting of points $\mathbf{r} = (r_1, r_2, \dots, r_n)$, $r_i \in D_{N-1}$, such that

$$N - 1 \geq r_1 > r_2 > \cdots > r_n \geq 0.$$

This transform is given by the kernel

$$\cos^{2,-}_{\mathbf{r}}(\mathbf{k}) = |S_n|^{-1/2} \det \left(\cos \frac{\pi(r_i + \frac{1}{2})k_j}{N} \right)_{i,j=1}^n, \quad (76)$$

where $\mathbf{r} \in \tilde{D}_N^n$ and $\mathbf{k} = (k_1, k_2, \dots, k_n)$, $k_i \in \{0, 1, 2, \dots, N-1\}$. The orthogonality relation for these kernels is

$$\langle \cos^{2,-}_{\mathbf{r}}(\mathbf{k}), \cos^{2,-}_{\mathbf{r}'}(\mathbf{k}) \rangle = |S_n| \sum_{\mathbf{k}\in\tilde{D}_{N-1}^n} c_{\mathbf{k}} \cos^{2,-}_{\mathbf{r}}(\mathbf{k}) \cos^{2,-}_{\mathbf{r}'}(\mathbf{k}) = \left(\frac{N}{2}\right)^n \delta_{\mathbf{r}\mathbf{r}'}, \quad (77)$$

where $c_{\mathbf{k}} = c_1 c_2 \cdots c_n$ and c_i are such as in formula (67).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in\tilde{D}_{N-1}^n} a_{\mathbf{r}} \cos^{2,-}_{\mathbf{r}}(\mathbf{k}), \quad \text{where} \quad a_{\mathbf{r}} = |S_n| \left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in\tilde{D}_{N-1}^n} c_{\mathbf{k}} f(\mathbf{k}) \cos^{2,-}_{\mathbf{r}}(\mathbf{k}). \quad (78)$$

The Plancherel formula is of the form

$$|S_n| \sum_{\mathbf{k}\in\tilde{D}_{N-1}^n} c_{\mathbf{k}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in\tilde{D}_{N-1}^n} |a_{\mathbf{r}}|^2.$$

AMDCT-3. This transform is given by the kernel

$$\cos^{3,-}_{\mathbf{r}}(\mathbf{k}) = |S_n|^{-1/2} \det \left(\cos \frac{\pi r_i (k_j + \frac{1}{2})}{N} \right)_{i,j=1}^n, \quad \mathbf{r} \in \tilde{D}_{N-1}^n, \quad k_j \in D_{N-1}. \quad (79)$$

The orthogonality relation for these kernels is

$$\langle \cos^{3,-}_{\mathbf{r}}(\mathbf{k}), \cos^{3,-}_{\mathbf{r}'}(\mathbf{k}) \rangle = |S_n| \sum_{\mathbf{k}\in\tilde{D}_{N-1}^n} \cos^{3,-}_{\mathbf{r}}(\mathbf{k}) \cos^{3,-}_{\mathbf{r}'}(\mathbf{k}) = h_{\mathbf{r}} \left(\frac{N}{2}\right)^n \delta_{\mathbf{r}\mathbf{r}'}, \quad (80)$$

where $h_{\mathbf{r}} = h_1 h_2 \cdots h_n$ and h_j are such as in formula (69).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in\tilde{D}_{N-1}^n} a_{\mathbf{r}} \cos^{3,-}_{\mathbf{r}}(\mathbf{k}), \quad \text{where} \quad a_{\mathbf{r}} = \frac{|S_n|}{h_{\mathbf{r}}} \left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in\tilde{D}_{N-1}^n} f(\mathbf{k}) \cos^{3,-}_{\mathbf{r}}(\mathbf{k}). \quad (81)$$

The Plancherel formula is of the form

$$|S_n| \sum_{\mathbf{k}\in \tilde{D}^n_{N-1}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in \tilde{D}^n_{N-1}} h_{\mathbf{r}} |a_{\mathbf{r}}|^2 .$$

AMDCT-4. This transform is given by the kernel

$$\cos^{4,-}_{\mathbf{r}}(\mathbf{k}) = |S_n|^{-1/2} \det \left(\cos \frac{\pi(r_i+\frac12)(k_j+\frac12)}{N} \right)^n_{i,j=1} , \tag{82}$$

where $\mathbf{r}\in \tilde{D}^n_{N-1}$ and $k_j \in D_{N-1}$. The orthogonality relation for these kernels is

$$\langle \cos^{4,-}_{\mathbf{r}}(\mathbf{k}), \cos^{4,-}_{\mathbf{r}'}(\mathbf{k})\rangle = |S_n| \sum_{\mathbf{k}\in \tilde{D}^n_{N-1}} \cos^{4,-}_{\mathbf{r}}(\mathbf{k}) \cos^{4,-}_{\mathbf{r}'}(\mathbf{k}) = \left(\frac{N}{2}\right)^n \delta_{\mathbf{r}\mathbf{r}'} . \tag{83}$$

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in \tilde{D}^n_{N-1}} a_{\mathbf{r}} \cos^{4,-}_{\mathbf{r}}(\mathbf{k}), \quad \text{where} \quad a_{\mathbf{r}} = |S_n| \left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in \tilde{D}^n_{N-1}} f(\mathbf{k}) \cos^{4,-}_{\mathbf{r}}(\mathbf{k}). \tag{84}$$

The Plancherel formula is

$$|S_n| \sum_{\mathbf{k}\in \tilde{D}^n_{N-1}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in \tilde{D}^n_{N-1}} |a_{\mathbf{r}}|^2 .$$

14. Other symmetric multivariate finite cosine transforms

To each of the finite cosine transforms DCT-1, DCT-2, DCT-3, DCT-4 there corresponds a symmetric multivariate finite cosine transform. We denote the corresponding transforms as SMDCT-1, SMDCT-2, SMDCT-3, SMDCT-4. Below we give these transforms without proof (proofs are the same as in the case of symmetric multivariate finite cosine transforms of section 11). We fix a positive integer N and use the notation $\breve{D}^n_N$ for the subset of the set $D^n_N \equiv D_N \times D_N \times \cdots \times D_N$ (n times) with $D_N = \{0,1,2,\dots,N\}$ consisting of points $\mathbf{r} = (r_1, r_2, \dots, r_n)$, $r_i \in \mathbb{Z}^{\ge 0}$, such that

$$N \ge r_1 \ge r_2 \ge \cdots \ge r_n \ge 0.$$

The set $\breve{D}^n_N$ is an extension of the set $\tilde{D}^n_N$ of the previous section by adding points which are invariant with respect of some elements of the permutation group S_n.

The set D^n_N is obtained by action by elements of the group S_n upon $\breve{D}^n_N$, that is, D^n_N coincides with the set $\{w\breve{D}^n_N; w\in S_n\}$. However, in $\{w\breve{D}^n_N; w\in S_n\}$, some points are met several times. Namely, a point $\mathbf{k}_0 \in \breve{D}^n_N$ is met $|S_{\mathbf{k}_0}|$ times in the set $\{w\breve{D}^n_N; w\in S_n\}$, where $|S_{\mathbf{k}_0}|$ is an order of the subgroup $S_{\mathbf{k}_0} \subset S_n$ consisting of elements $w\in S_n$ leaving $\mathbf{k}_0$ invariant.

SMDCT-1. This transform is given by the kernel

$$\cos^{+}_{\mathbf{r}}(\mathbf{k}) \equiv \cos^{1,+}_{\mathbf{r}}(\mathbf{k}) = |S_n|^{-1/2} \det{}^{+} \left(\cos \frac{\pi r_i k_j}{N} \right)^n_{i,j=1} , \tag{85}$$

where $\mathbf{k} = (k_1, k_2, \dots, k_n)$, $k_i \in \{0,1,2,\dots,N\}$. The orthogonality relation for these kernels is

$$\langle \cos^{+}_{\mathbf{r}}(\mathbf{k}), \cos^{+}_{\mathbf{r}'}(\mathbf{k})\rangle = |S_n| \sum_{\mathbf{k}\in \breve{D}^n_N} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} \cos^{+}_{\mathbf{r}}(\mathbf{k}) \cos^{+}_{\mathbf{r}'}(\mathbf{k})$$

$$= h_{\mathbf{r}} \left(\frac{N}{2}\right)^n |S_{\mathbf{r}}| \delta_{\mathbf{r}\mathbf{r}'}, \tag{86}$$

where $S_{\mathbf{r}}$ is the subgroup of S_n consisting of elements leaving $\mathbf{r}$ invariant,

$$c_{\mathbf{k}} = c_1 c_2 \cdots c_n, \quad h_{\mathbf{k}} = h_1 h_2 \cdots h_n,$$

and c_i and h_j are such as in formula (65).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r} \in \breve{D}_N^n} a_{\mathbf{r}} \cos_{\mathbf{r}}^+(\mathbf{k}), \tag{87}$$

where

$$a_{\mathbf{r}} = \frac{|S_n|}{h_{\mathbf{r}} |S_{\mathbf{r}}|} \left(\frac{2}{N}\right)^n \sum_{\mathbf{k} \in \breve{D}_N^n} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} f(\mathbf{k}) \cos_{\mathbf{r}}^+(\mathbf{k}).$$

The Plancherel formula is

$$|S_n| \sum_{\mathbf{k} \in \breve{D}_N^n} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r} \in \breve{D}_N^n} h_{\mathbf{r}} |S_{\mathbf{r}}| |a_{\mathbf{r}}|^2.$$

This transform is in fact a variation of the symmetric multivariate discrete cosine transforms from section 11.

SMDCT-2. This transform is given by the kernel

$$\cos_{\mathbf{r}}^{2,+}(\mathbf{k}) = |S_n|^{-1/2} \det{}^+ \left(\cos \frac{\pi (r_i + \frac{1}{2}) k_j}{N}\right)_{i,j=1}^n, \quad \mathbf{r} \in \breve{D}_{N-1}^n, \tag{88}$$

where $\breve{D}_{N-1}^n$ is the set $\breve{D}_N^n$ with N replaced by $N-1$ and $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, $k_i \in \{0, 1, 2, \ldots, N-1\}$. The orthogonality relation for these kernels is

$$\langle \cos_{\mathbf{r}}^{2,+}(\mathbf{k}), \cos_{\mathbf{r}'}^{2,+}(\mathbf{k}) \rangle = |S_n| \sum_{\mathbf{k} \in \breve{D}_{N-1}^n} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} \cos_{\mathbf{r}}^{2,+}(\mathbf{k}) \cos_{\mathbf{r}'}^{2,+}(\mathbf{k})$$

$$= \left(\frac{N}{2}\right)^n |S_{\mathbf{r}}| \delta_{\mathbf{r}\mathbf{r}'}, \tag{89}$$

where $c_{\mathbf{k}} = c_1 c_2 \cdots c_n$ and c_j are such as in (67).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r} \in \breve{D}_{N-1}^n} a_{\mathbf{r}} \cos_{\mathbf{r}}^{2,+}(\mathbf{k}), \tag{90}$$

where

$$a_{\mathbf{r}} = \frac{|S_n|}{|S_{\mathbf{r}}|} \left(\frac{2}{N}\right)^n \sum_{\mathbf{k} \in \breve{D}_{N-1}^n} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} f(\mathbf{k}) \cos_{\mathbf{r}}^{2,+}(\mathbf{k}).$$

The Plancherel formula is of the form

$$|S_n| \sum_{\mathbf{k} \in \breve{D}_{N-1}^n} |S_{\mathbf{k}}|^{-1} c_{\mathbf{k}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r} \in \breve{D}_{N-1}^n} |S_{\mathbf{r}}| |a_{\mathbf{r}}|^2.$$

SMDCT-3. This transform is given by the kernel

$$\cos_{\mathbf{r}}^{3,+}(\mathbf{k}) = |S_n|^{-1/2} \det{}^+ \left(\cos \frac{\pi r_i (k_j + \frac{1}{2})}{N}\right)_{i,j=1}^n, \tag{91}$$

where $\mathbf{r} \in \breve{D}^n_{N-1}$. The orthogonality relation for these kernels is

$$\langle \cos^{3,+}_{\mathbf{r}}(\mathbf{k}), \cos^{3,+}_{\mathbf{r}'}(\mathbf{k})\rangle = |S_n| \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} \cos^{3,+}_{\mathbf{r}}(\mathbf{k}) \cos^{3,+}_{\mathbf{r}'}(\mathbf{k})$$

$$= h_{\mathbf{r}} \left(\frac{N}{2}\right)^n |S_{\mathbf{r}}| \delta_{\mathbf{r}\mathbf{r}'}, \tag{92}$$

where $h_{\mathbf{r}} = h_1 h_2 \cdots h_n$ and h_i are such as in formula (69).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in \breve{D}^n_{N-1}} a_{\mathbf{r}} \cos^{3,+}_{\mathbf{r}}(\mathbf{k}), \tag{93}$$

where

$$a_{\mathbf{r}} = \frac{|S_n|}{h_{\mathbf{r}}|S_{\mathbf{r}}|} \left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} f(\mathbf{k}) \cos^{3,+}_{\mathbf{r}}(\mathbf{k}).$$

The Plancherel formula is of the form

$$|S_n| \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in \breve{D}^n_{N-1}} h_{\mathbf{r}} |S_{\mathbf{r}}| |a_{\mathbf{r}}|^2.$$

SMDCT-4. This transform is given by the kernel

$$\cos^{4,+}_{\mathbf{r}}(\mathbf{k}) = |S_n|^{-1/2} \mathrm{det}^+ \left(\cos \frac{\pi(r_i+\frac12)(k_j+\frac12)}{N}\right)_{i,j=1}^n, \tag{94}$$

where $\mathbf{r} \in \breve{D}^n_{N-1}$. The orthogonality relation for these kernels is

$$\langle \cos^{4,+}_{\mathbf{r}}(\mathbf{k}), \cos^{4,+}_{\mathbf{r}'}(\mathbf{k})\rangle = |S_n| \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} \cos^{4,+}_{\mathbf{r}}(\mathbf{k}) \cos^{4,+}_{\mathbf{r}'}(\mathbf{k})$$

$$= \left(\frac{N}{2}\right)^n |S_{\mathbf{r}}| \delta_{\mathbf{r}\mathbf{r}'}. \tag{95}$$

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in \breve{D}^n_{N-1}} a_{\mathbf{r}} \cos^{4,+}_{\mathbf{r}}(\mathbf{k}), \tag{96}$$

where

$$a_{\mathbf{r}} = \left(\frac{2}{N}\right)^n \frac{|S_n|}{|S_{\mathbf{r}}|} \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} f(\mathbf{k}) \cos^{4,+}_{\mathbf{r}}(\mathbf{k}).$$

The Plancherel formula is

$$|S_n| \sum_{\mathbf{k}\in \breve{D}^n_{N-1}} |S_{\mathbf{k}}|^{-1} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in \breve{D}^n_{N-1}} |S_{\mathbf{r}}| |a_{\mathbf{r}}|^2.$$

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